ASYMPTOTIC EXPRESSIONS OF EIGENVALUES AND FUNDAMENTAL SOLUTIONS OF A DISCONTINUOUS FOURTH-ORDER BOUNDARY VALUE PROBLEM

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Abstract. In the present paper, we deal with a fourth-order boundary value problem problem with eigenparameter dependent boundary conditions and transmission conditions at a interior point. A self-adjoint linear operator A is defined in a suitable Hilbert space H such that the eigenvalues of such a problem coincide with those of A. We obtain asymptotic formulae for its eigenvalues and fundamental solutions. Our applications possess a number of interesting properties for studying in boundary value problems which we state in this paper.

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1. Introduction

It is well-known that many topics in mathematical physics require the investigation of eigenvalues and eigenfunctions of Sturm-Liouville type boundary value problems. In recent years, more and more researchers are interested in the discontinuous Sturm-Liouville problem (see [1-6]). Various physics applications of this kind problem are found in many literatures, including some boundary value problem with transmission conditions that arise in the theory of heat and mass transfer (see [8,9]). The literature on such results is voluminous and we refer to [1-11].

Fourth-order discontinuous boundary value problems with eigen-dependent boundary conditions and with two supplementary transmission conditions at the point of discontinuity have been investigated in [12,13]. Note that discontinuous Sturm-Liouville problems with eigen-dependent boundary conditions and with four supplementary transmission conditions at the points of discontinuity have been investigated in [3].

In this study, we shall consider a fourth-order differential equation

$$Lu := \left(a\left(x\right)u''\left(x\right)\right)'' + q(x)u(x) = \lambda u(x) \tag{1.1}$$

on $I = [-1, 0) \cup (0, 1]$, with boundary conditions at $x = -1$

$$L_1 u := u\left(-1\right) = 0, \tag{1.2}$$

$$L_2 u := \beta_1 u'(-1) + \beta_2 u''\left(-1\right) = 0, \tag{1.3}$$

with the six transmission conditions at the points of discontinuity $x = 0$,

$$L_3 u := u(0+) - u(0-) = 0, \tag{1.4}$$
$$L_4 u := u'(0+) - u'(0-) = 0, \tag{1.5}$$
$$L_5 u := u''(0+) - u''(0-) + \lambda\delta_1 u'(0-) = 0, \tag{1.6}$$
$$L_6 u := u'''(0+) - u'''(0-) + \lambda\delta_2 u(0-) = 0, \tag{1.7}$$

and the eigen-dependent boundary conditions at $x = 1$

$$L_7 u := \lambda u(1) + u'''(1) = 0, \tag{1.8}$$
$$L_8 u := \lambda u'(1) + u''(1) = 0, \tag{1.9}$$

where $a(x) = a_1^4$, for $x \in [-1, 0)$, $a(x) = a_2^4$, for $x \in (0, 1]$, $a_1 > 0$ and $a_2 > 0$ are given real numbers, $q(x)$ is a given real-valued function continuous in $[-1, 0) \cup (0, 1]$ and has a finite limit $q(0\pm) = \lim_{x\to 0} \pm q(x)$; λ is a complex eigenvalue parameter; β_i, δ_i $(i = 1, 2)$ are real numbers and $|\beta_1| + |\beta_2| \neq 0$, $|\delta_1| + |\delta_2| \neq 0$.

2. Preliminaries

Firstly we define the inner product in L^2 for every $f, g \in L^2(I)$ as

$$\langle f, g\rangle_1 = \frac{1}{a_1^4}\int_{-1}^{0} f_1\overline{g_1}dx + \frac{1}{a_2^4}\int_0^1 f_2\overline{g_2}dx,$$

where $f_1(x) = f(x)\big|_{[-1,0)}$, $f_2(x) = f(x)\big|_{(0,1]}$. It is easy to see that $\left(L^2(I), [\cdot,\cdot]\right)$ is a Hilbert space. Now we define the inner product in the direct sum of spaces $L^2(I) \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}_{\delta_1} \oplus \mathbb{C}_{\delta_2}$ by

$$[F, G] := \langle f, g\rangle_1 + \langle h_1, k_1\rangle + \langle h_2, k_2\rangle + \langle h_3, k_3\rangle + \langle h_4, k_4\rangle$$

for

$$F := (f, h_1, h_2, h_3, h_4), G := (g, k_1, k_2, k_3, k_4) \in L^2(I) \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}_{\delta_1} \oplus \mathbb{C}_{\delta_2}.$$

Then $Z := \left(L^2(I) \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}_{\delta_1} \oplus \mathbb{C}_{\delta_2}, [\cdot,\cdot]\right)$ is the direct sum of modified Krein spaces. A fundamental symmetry on the Krein space is given by

$$J := \begin{bmatrix} J_0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & sgn\delta_1 & 0 \\ 0 & 0 & 0 & 0 & sgn\delta_2 \end{bmatrix},$$

where $J_0 : L^2(I) \to L^2(I)$ is defined by $(J_0 f)(x) = f(x)$. We define a linear operator A in Z by the domain of definition

$$D(A) := (f,\ h_1,\ h_2,\ h_3,\ h_4) \in Z \mid f_1^{(i)} \in AC_{loc}((-1, 0)),\ f_2^{(i)} \in AC_{loc}((0, 1)),\ i = \overline{0, 3},$$
$$Lf \in L^2(I),\ L_k f = 0,\ k = \overline{1, 6},\ h_1 = f(1),\ h_2 = f'(1),\ h_3 = -\delta_1 f'(0),\ h_4 = -\delta_2 f(0),$$
$$AF = \left(Lf,\ -f'''(1),\ -f''(1),\ f''(0+) - f''(0-),\ f'''(0+) - f'''(0-)\right),$$
$$F = \left(f,\ f(1),\ f'(1), -\delta_1 f'(0),\ -\delta_2 f(0)\right) \in D(A).$$

Consequently, the considered problem (1.1)-(1.9) can be rewritten in operator form as

$$AF = \lambda F,$$

i.e., the problem (1.1)-(1.9) can be considered as the eigenvalue problem for the operator A. Then, we can write the following conclusions:

Theorem 2.1. *The eigenvalues and eigenfunctions of the problem (1.1)-(1.9) are defined as the eigenvalues and the first components of the corresponding eigenelements of the operator A respectively.*

Theorem 2.2. *The operator A is self-adjoint in Krein space Z (cf. Theorem 2.2 of* [10]*).*

3. Fundamental Solutions

Lemma 3.1. *Let the real-valued function $q(x)$ be continuous in $[-1,1]$ and $f_i(\lambda)$ $(i = 1,4)$ are given entire functions. Then for any $\lambda \in \mathbb{C}$ the equation*

$$\left(a(x)\,u''(x)\right)'' + q(x)u(x) = \lambda u(x), \quad x \in I$$

has a unique solution $u = u(x,\lambda)$ such that

$$u(-1) = f_1(\lambda),\; u^{'}(-1) = f_2(\lambda),\; u''(-1) = f_3(\lambda),\; u'''(-1) = f_4(\lambda)$$
$$\left(or\; u(1) = f_1(\lambda),\; u^{'}(1) = f_2(\lambda),\; u''(1) = f_3(\lambda),\; u'''(1) = f_4(\lambda)\right).$$

and for each $x \in [-1,1]$, $u(x,\lambda)$ is an entire function of λ.

Proof. Let $\phi_{11}(x,\lambda)$ be the solution of Eq. (1.1) on $[-1,0)$ which satisfies the initial conditions

$$\phi_{11}(-1) = 0,\; \phi'_{11}(-1) = \phi''_{11}(-1) = 0,$$
$$\phi'''_{11}(-1) = -1.$$

By virtue of Lemma 3.1, after defining this solution, we may define the solution $\phi_{12}(x,\lambda)$ of Eq. (1.1) on $(0,1]$ by means of the solution $\phi_{11}(x,\lambda)$ by the initial conditions

$$\phi_{12}(0) = \phi_{11}(0),\; \phi'_{12}(0) = \phi'_{11}(0),\; \phi''_{12}(0) = \phi''_{11}(0) - \lambda\delta_1\phi'_{11}(0),$$
$$\phi'''_{12}(0) = \phi'''_{11}(0) - \lambda\delta_2\phi_{11}(0). \tag{3.1}$$

After defining this solution, we may define the solution $\phi_{21}(x,\lambda)$ of Eq. (1.1) on $[-1,0)$ which satisfies the initial conditions

$$\phi_{21}(-1) = 0,\; \phi'_{21}(-1) = \beta_2,\; \phi''_{21}(-1) = -\beta_1,\; \phi'''_{21}(-1) = 0. \tag{3.2}$$

After defining this solution, we may define the solution $\phi_{22}(x,\lambda)$ of Eq. (1.1) on $(0,1]$ by means of the solution $\phi_{21}(x,\lambda)$ by the initial conditions

$$\phi_{22}(0) = \phi_{21}(0),\; \phi'_{22}(0) = \phi'_{21}(0),\; \phi''_{22}(0) = \phi''_{21}(0) - \lambda\delta_1\phi'_{21}(0),$$
$$\phi'''_{22}(0) = \phi'''_{21}(0) - \lambda\delta_2\phi_{21}(0). \tag{3.3}$$

Analogically we shall define the solutions $\chi_{11}(x,\lambda)$ and $\chi_{12}(x,\lambda)$ by the initial conditions

$$\chi_{12}(1) = -1,\; \chi'_{12}(1) = \chi''_{12}(1) = 0, \chi'''_{12}(1) = \lambda,\; \chi_{11}(0) = \chi_{12}(0),$$
$$\chi'_{11}(0) = \chi'_{12}(0),\; \chi''_{11}(0) = \chi''_{12}(0) + \lambda\delta_1\chi'_{12}(0), \tag{3.4}$$
$$\chi'''_{11}(0) = \chi'''_{12}(0) + \lambda\delta_2\chi_{12}(0).$$

Moreover, we shall define the solutions $\chi_{21}(x,\lambda)$ and $\chi_{22}(x,\lambda)$ by the initial conditions

$$\begin{gathered}\chi_{22}(1)=0,\ \chi_{22}'(1)=-1,\ \chi_{22}''(1)=\lambda,\ \chi_{22}'''(1)=0,\ \chi_{21}(0)=\chi_{22}(0),\\ \chi_{21}'(0)=\chi_{22}'(0),\ \chi_{21}''(0)=\chi_{22}''(0)+\lambda\delta_1\chi_{22}'(0),\\ \chi_{21}'''(0)=\chi_{22}'''(0)+\lambda\delta_2\chi_{22}(0).\end{gathered} \tag{3.5}$$

Let us consider the Wronskians

$$W_1(\lambda):=\begin{vmatrix}\phi_{11}(x,\lambda) & \phi_{21}(x,\lambda) & \chi_{11}(x,\lambda) & \chi_{21}(x,\lambda)\\ \phi_{11}'(x,\lambda) & \phi_{21}'(x,\lambda) & \chi_{11}'(x,\lambda) & \chi_{21}'(x,\lambda)\\ \phi_{11}''(x,\lambda) & \phi_{21}''(x,\lambda) & \chi_{11}''(x,\lambda) & \chi_{21}''(x,\lambda)\\ \phi_{11}'''(x,\lambda) & \phi_{21}'''(x,\lambda) & \chi_{11}'''(x,\lambda) & \chi_{21}'''(x,\lambda)\end{vmatrix}$$

and

$$W_2(\lambda):=\begin{vmatrix}\phi_{12}(x,\lambda) & \phi_{22}(x,\lambda) & \chi_{12}(x,\lambda) & \chi_{22}(x,\lambda)\\ \phi_{12}'(x,\lambda) & \phi_{22}'(x,\lambda) & \chi_{12}'(x,\lambda) & \chi_{22}'(x,\lambda)\\ \phi_{12}''(x,\lambda) & \phi_{22}''(x,\lambda) & \chi_{12}''(x,\lambda) & \chi_{22}''(x,\lambda)\\ \phi_{12}'''(x,\lambda) & \phi_{22}'''(x,\lambda) & \chi_{12}'''(x,\lambda) & \chi_{22}'''(x,\lambda)\end{vmatrix},$$

which are independent of x and entire functions. This sort of calculation gives $W_1(\lambda)=W_2(\lambda)$. Now we may introduce in consideration the characteristic function $W(\lambda)$ as $W(\lambda)=W_1(\lambda)$. □

Theorem 3.2. *The eigenvalues of the problem (1.1)-(1.9) are the zeros of the function* $W(\lambda)$.

Proof. Let $W(\lambda)=0$. Then the functions $\phi_{11}(x,\lambda)$, $\phi_{21}(x,\lambda)$ and $\chi_{11}(x,\lambda)$, $\chi_{21}(x,\lambda)$ are linearly dependent, i.e.,

$$k_1\phi_{11}(x,\lambda)+k_2\phi_{21}(x,\lambda)+k_3\chi_{11}(x,\lambda)+k_4\chi_{21}(x,\lambda)=0$$

for some $k_1\neq 0$ or $k_2\neq 0$ and $k_3\neq 0$ or $k_4\neq 0$. From this, it follows that $k_3\chi_{11}(x,\lambda)+k_4\chi_{21}(x,\lambda)$ satisfies the boundary conditions (1.2)-(1.3). Therefore

$$\begin{cases} k_3\chi_{11}(x,\lambda)+k_4\chi_{21}(x,\lambda), & x\in[-1,0),\\ k_3\chi_{12}(x,\lambda)+k_4\chi_{22}(x,\lambda), & x\in(0,1]\end{cases}$$

is an eigenfunction of the problem (1.1)-(1.9) corresponding to eigenvalue λ.

Now we let $u(x)$ be any eigenfunction corresponding to eigenvalue λ, but $W(\lambda)\neq 0$. Then the functions ϕ_{11}, ϕ_{21}, χ_{11}, χ_{21} would be linearly independent on $(0,1]$. Therefore $u(x)$ may be represented as

$$u(x)=\begin{cases} c_1\phi_{11}(x,\lambda)+c_2\phi_{21}(x,\lambda)+c_3\chi_{11}(x,\lambda)+c_4\chi_{21}(x,\lambda), & x\in[-1,0);\\ c_5\phi_{12}(x,\lambda)+c_6\phi_{22}(x,\lambda)+c_7\chi_{12}(x,\lambda)+c_8\chi_{22}(x,\lambda), & x\in(0,1],\end{cases}$$

where at least one of the constants c_1, c_2, c_3, c_4, c_5, c_6, c_7 and c_8 is not zero. Considering the equations

$$L_\upsilon(u(x))=0,\ \ \upsilon=\overline{1,8} \tag{3.6}$$

as a system of linear equations of the variables c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8 and taking (3.1)-(3.5) into account, it follows that the determinant of this

system is

$$\begin{vmatrix}
0 & 0 & L_1\chi_{11} & L_1\chi_{21} & 0 & 0 & 0 & 0 \\
0 & 0 & L_2\chi_{11} & L_2\chi_{21} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & L_3\phi_{12} & L_3\phi_{22} & 0 & 0 \\
0 & 0 & 0 & 0 & L_4\phi_{12} & L_4\phi_{22} & 0 & 0 \\
-\phi_{12}(0) & -\phi_{22}(0) & -\chi_{12}(0) & -\chi_{22}(0) & \phi_{12}(0) & \phi_{22}(0) & \chi_{12}(0) & \chi_{22}(0) \\
-\phi'_{12}(0) & -\phi'_{22}(0) & -\chi'_{12}(0) & -\chi'_{22}(0) & \phi'_{12}(0) & \phi'_{22}(0) & \chi'_{12}(0) & \chi'_{22}(0) \\
-\phi''_{12}(0) & -\phi''_{22}(0) & -\chi''_{12}(0) & -\chi''_{22}(0) & \phi''_{12}(0) & \phi''_{22}(0) & \chi''_{12}(0) & \chi''_{22}(0) \\
-\phi'''_{12}(0) & -\phi'''_{22}(0) & -\chi'''_{12}(0) & -\chi'''_{22}(0) & \phi'''_{12}(0) & \phi'''_{22}(0) & \chi'''_{12}(0) & \chi'''_{22}(0)
\end{vmatrix}$$
$$= -W(\lambda)^3 \neq 0.$$

Therefore, the system (3.6) has only the trivial solution $c_i = 0$ $\left(i = \overline{1,8}\right)$. Thus we get a contradiction, which completes the proof. $\square$

4. **Asymptotic formulae for eigenvalues and fundamental solutions**

We start by proving some lemmas.

Lemma 4.1. *Let $\phi(x,\lambda)$ be the solution of Eq. (1.1) defined in Section 3, and let $\lambda = s^4$, $s = \sigma + it$. Then the following integral equations hold for $k = \overline{0,3}$:*

$$\begin{aligned}
&\frac{d^k}{dx^k}\phi_{11}(x,\lambda) \\
&= \frac{a_1^3}{2s^3}\frac{d^k}{dx^k}\sin\frac{s(x+1)}{a_1} + \frac{a_1^3}{4s^3}\frac{d^k}{dx^k}e^{\frac{s(x+1)}{a_1}} + \frac{a_1^3}{4s^3}\frac{d^k}{dx^k}e^{-\frac{s(x+1)}{a_1}} \\
&+\frac{a_1^3}{2s^3}\int_{-1}^{x}\frac{d^k}{dx^k}\left(\sin\frac{s(x-y)}{a_1} - e^{\frac{s(x-y)}{a_1}} + e^{-\frac{s(x-y)}{a_1}}\right)q(y)\,\phi_{11}(y,\lambda)\,dy.
\end{aligned}\tag{4.1}$$

$$\begin{aligned}
&\frac{d^k}{dx^k}\phi_{12}(x,\lambda) \\
&= \left(\frac{\phi_{12}(0)}{2} - \frac{a_2^2\phi''_{12}(0)}{2s^2}\right)\frac{d^k}{dx^k}\cos\frac{sx}{a_2} + \left(\frac{a_2\phi'_{12}(0)}{2s} - \frac{a_2^3\phi'''_{12}(0)}{2s^3}\right) \\
&\times\frac{d^k}{dx^k}\sin\frac{sx}{a_2} + \left(\frac{\phi_{12}(0)}{4} + \frac{a_2\phi'_{12}(0)}{4s} + \frac{a_2^2\phi''_{12}(0)}{4s^2} + \frac{a_2^3\phi'''_{12}(0)}{4s^3}\right) \\
&\times\frac{d^k}{dx^k}e^{\frac{sx}{a_2}} + \left(\frac{\phi_{12}(0)}{4} - \frac{a_2\phi'_{12}(0)}{4s} + \frac{a_2^2\phi''_{12}(0)}{4s^2} - \frac{a_2^3\phi'''_{12}(0)}{4s^3}\right)\frac{d^k}{dx^k}e^{-\frac{sx}{a_2}} \\
&+\frac{a_2^3}{2s^3}\int_{0}^{x}\frac{d^k}{dx^k}\left(\sin\frac{s(x-y)}{a_2} - e^{\frac{s(x-y)}{a_2}} + e^{-\frac{s(x-y)}{a_2}}\right)q(y)\,\phi_{12}(y,\lambda)\,dy.
\end{aligned}\tag{4.2}$$

$$\frac{d^k}{dx^k}\phi_{21}(x,\lambda)$$
$$= \frac{\beta_1 a_1^2}{2s^2}\frac{d^k}{dx^k}\cos\frac{s(x+1)}{a_1} + \frac{\beta_2 a_1}{2s}\frac{d^k}{dx^k}\sin\frac{s(x+1)}{a_1}$$
$$+\left(\frac{\beta_2 a_1}{4s} - \frac{\beta_1 a_1^2}{4s^2}\right)\frac{d^k}{dx^k}e^{\frac{s(x+1)}{a_1}} - \left(\frac{\beta_2 a_1}{4s} + \frac{\beta_1 a_1^2}{4s^2}\right)\frac{d^k}{dx^k}e^{-\frac{s(x+1)}{a_1}}$$
$$+\frac{a_1^3}{2s^3}\int_{-1}^{x}\frac{d^k}{dx^k}\left(\sin\frac{s(x-y)}{a_1} - e^{\frac{s(x-y)}{a_1}} + e^{-\frac{s(x-y)}{a_1}}\right)q(y)\,\phi_{21}(y,\lambda)\,dy. \tag{4.3}$$

$$\frac{d^k}{dx^k}\phi_{22}(x,\lambda)$$
$$= \left(\frac{\phi_{22}(0)}{2} - \frac{a_2^2\phi_{22}''(0)}{2s^2}\right)\frac{d^k}{dx^k}\cos\frac{sx}{a_2} + \left(\frac{a_2\phi_{22}'(0)}{2s} - \frac{a_2^3\phi_{22}'''(0)}{2s^3}\right)$$
$$\times\frac{d^k}{dx^k}\sin\frac{sx}{a_2} + \left(\frac{\phi_{22}(0)}{4} + \frac{a_2\phi_{22}'(0)}{4s} + \frac{a_2^2\phi_{22}''(0)}{4s^2} + \frac{a_2^3\phi_{22}'''(0)}{4s^3}\right)$$
$$\times\frac{d^k}{dx^k}e^{\frac{sx}{a_2}} + \left(\frac{\phi_{22}(0)}{4} - \frac{a_2\phi_{22}'(0)}{4s} + \frac{a_2^2\phi_{22}''(0)}{4s^2} - \frac{a_2^3\phi_{22}'''(0)}{4s^3}\right)\frac{d^k}{dx^k}e^{-\frac{sx}{a_2}}$$
$$+\frac{a_2^3}{2s^3}\int_{0}^{x}\frac{d^k}{dx^k}\left(\sin\frac{s(x-y)}{a_2} - e^{\frac{s(x-y)}{a_2}} + e^{-\frac{s(x-y)}{a_2}}\right)q(y)\,\phi_{22}(y,\lambda)\,dy. \tag{4.4}$$

Proof. Regard $\phi_{11}(x,\lambda)$ as the solution of the following non-homogeneous Cauchy problem:

$$\begin{cases} -\left(a(x)\,u''(x)\right)'' - s^4 u(x) = q(x)\,\phi_{11}(x,\lambda), \\ \phi_{11}(-1,\lambda) = 1,\ \phi_{11}'(-1,\lambda) = 0, \\ \phi_{11}''(-1,\lambda) = 0,\ \phi_{11}'''(-1,\lambda) = 0. \end{cases}$$

Using the method of constant changing, $\phi_{11}(x,\lambda)$ satisfies

$$\phi_{11}(x,\lambda) = \frac{a_1^3}{2s^3}\sin\frac{s(x+1)}{a_1} + \frac{a_1^3}{4s^3}e^{\frac{s(x+1)}{a_1}} + \frac{a_1^3}{4s^3}e^{-\frac{s(x+1)}{a_1}} \tag{4.1}$$
$$+\frac{a_1^3}{2s^3}\int_{-1}^{x}\left(\sin\frac{s(x-y)}{a_1} - e^{\frac{s(x-y)}{a_1}} + e^{-\frac{s(x-y)}{a_1}}\right)q(y)\,\phi_{11}(y,\lambda)\,dy.$$

Then differentiating it with respect to x, we have (4.1). The proof for (4.2), (4.3) and (4.4) is similar. □

Lemma 4.2. *Let $\lambda = s^4$, $s = \sigma + it$. Then the following integral equations hold for $k = \overline{0,3}$:*

$$\frac{d^k}{dx^k}\phi_{11}(x,\lambda) = O\left(|s|^{k-1}\,e^{|s|\frac{(x+1)}{a_1}}\right). \tag{4.5}$$

$$\begin{aligned}
&\frac{d^k}{dx^k}\phi_{12}(x,\lambda)\\
&= \frac{a_2^2 s^2 \delta_1 \phi_{11}'(0)}{2}\frac{d^k}{dx^k}\cos\frac{sx}{a_2} + \frac{a_2^3 s \delta_2 \phi_{11}(0)}{2}\frac{d^k}{dx^k}\sin\frac{sx}{a_2}\\
&-\frac{a_2^2 s^2 \delta_1 \phi_{11}'(0)}{4}\frac{d^k}{dx^k}\left(e^{\frac{sx}{a_2}}+e^{-\frac{sx}{a_2}}\right) - \frac{a_2^3 s \delta_2 \phi_{11}(0)}{4}\frac{d^k}{dx^k}\left(e^{\frac{sx}{a_2}}-e^{-\frac{sx}{a_2}}\right)\\
&+O\left(e^{|s|^k\left(\frac{a_1x+a_2}{a_1a_2}\right)}\right).
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
&\frac{d^k}{dx^k}\phi_{21}(x,\lambda)\\
&= \frac{\beta_2 a_1}{2s}\frac{d^k}{dx^k}\sin\frac{s(x+1)}{a_1} + \frac{\beta_2 a_1}{4s}\frac{d^k}{dx^k}\left(e^{\frac{s(x+1)}{a_1}}-e^{-\frac{s(x+1)}{a_1}}\right)+O\left(|s|^{k-2}e^{|s|\frac{x+1}{a_1}}\right).
\end{aligned}$$

$$\begin{aligned}
&\frac{d^k}{dx^k}\phi_{22}(x,\lambda)\\
&= \frac{a_2^2 s^2 \delta_1 \phi_{21}'(0)}{2}\frac{d^k}{dx^k}\cos\frac{sx}{a_2} + \frac{a_2^3 s \delta_2 \phi_{21}(0)}{2}\frac{d^k}{dx^k}\sin\frac{sx}{a_2}\\
&-\frac{a_2^2 s^2 \delta_1 \phi_{21}'(0)}{4}\frac{d^k}{dx^k}\left(e^{\frac{sx}{a_2}}+e^{-\frac{sx}{a_2}}\right) - \frac{a_2^3 s \delta_2 \phi_{21}(0)}{4}\frac{d^k}{dx^k}\left(e^{\frac{sx}{a_2}}-e^{-\frac{sx}{a_2}}\right)\\
&+O\left(e^{|s|^{k-1}\left(\frac{a_1x+a_2}{a_1a_2}\right)}\right).
\end{aligned}$$

Each of these asymptotic formulae holds uniformly for x as $|\lambda| \to \infty$.

Proof. Let

$$F_{11}(x,\lambda) = e^{-|s|\frac{x+1}{a_1}}\phi_{11}(x,\lambda).$$

It is easy to see that $F_{11}(x,\lambda)$ is bounded. Therefore $\phi_{11}(x,\lambda) = O(e)$. Substituting it into (4.1) and differentiating it with respect to x for $k=\overline{0,3}$, we obtain (4.5). According to transmission conditions (1.4)-(1.7) as $|\lambda|\to\infty$, we get

$$\phi_{12}(0)=\phi_{11}(0),\ \phi_{12}'(0)=\phi_{11}'(0),\ \phi_{12}''(0)=-s^4\delta_1\phi_{11}'(0),$$
$$\phi_{12}'''(0)=-s^4\delta_2\phi_{11}(0).$$

Substituting these asymptotic formulae into (4.2) for $k=0$, we obtain

$$\begin{aligned}
\phi_{12}(x,\lambda) &= \frac{a_2^2 s^2 \delta_1 \phi_{11}'(0)}{2}\cos\frac{sx}{a_2} + \frac{a_2^3 s \delta_2 \phi_{11}(0)}{2}\sin\frac{sx}{a_2}\\
&-\frac{a_2^2 s^2 \delta_1 \phi_{11}'(0)}{4}\left(e^{\frac{sx}{a_2}}+e^{-\frac{sx}{a_2}}\right) - \frac{a_2^3 s \delta_2 \phi_{11}(0)}{4}\left(e^{\frac{sx}{a_2}}-e^{-\frac{sx}{a_2}}\right)\\
&+\frac{a_2^3}{2s^3}\int_0^x\left(\sin\frac{s(x-y)}{a_2} - e^{\frac{s(x-y)}{a_2}} + e^{-\frac{s(x-y)}{a_2}}\right)q(y)\,\phi_{12}(y,\lambda)\,dy\\
&+O\left(e^{|s|\left(\frac{a_1x+a_2}{a_1a_2}\right)}\right).
\end{aligned} \tag{4.7}$$

Multiplying through by $|s|^{-3} e^{-|s|\left(\frac{a_1x+a_2}{a_1a_2}\right)}$, and denoting

$$F_{12}(x,\lambda) := O\left(|s|^{-3} e^{-|s|\left(\frac{a_1x+a_2}{a_1a_2}\right)}\right)\phi_{12}(x,\lambda).$$

Denoting $M := \max_{x\in[0,1]} |F_{12}(x,\lambda)|$ from the last formula, it follows that

$$M(\lambda) \le \frac{M(\lambda)}{2|s|^3}\int_0^x q(y)\,dy + M_0$$

for some $M_0 > 0$. From this, it follows that $M(\lambda) = O(1)$ as $|\lambda| \to \infty$, so

$$\phi_{12}(x,\lambda) = O\left(|s|^3 e^{|s|\left(\frac{a_1x+a_2}{a_1a_2}\right)}\right).$$

Substituting this back into the integral on the right side of (4.7) yields (4.6) for $k = 0$. The other cases may be considered analogically. □

Similarly one can establish the following lemma. for $\chi_{ij}(x,\lambda)$ $(i = 1,2, j = 1,2)$.

Lemma 4.3. *Let $\lambda = s^4$, $s = \sigma + it$. Then the following integral equations hold for $k = \overline{0,3}$:*

$$\begin{aligned}&\frac{d^k}{dx^k}\chi_{11}(x,\lambda)\\ &= -\frac{a_1^2 s^2\delta_1\chi_{12}'(0)}{2}\frac{d^k}{dx^k}\cos\frac{sx}{a_1} + \frac{a_1^3 s\delta_2\chi_{12}(0)}{2}\frac{d^k}{dx^k}\sin\frac{sx}{a_1}\\ &+\frac{a_1^2 s^2\delta_1\chi_{12}'(0)}{4}\frac{d^k}{dx^k}\left(e^{\frac{sx}{a_1}} + e^{-\frac{sx}{a_1}}\right)\\ &+\frac{a_1^3 s\delta_2\chi_{12}(0)}{4}\frac{d^k}{dx^k}\left(e^{\frac{sx}{a_1}} - e^{-\frac{sx}{a_1}}\right) + O\left(|s|^{k+1} e^{|s|\left(\frac{a_1-a_2x}{a_1a_2}\right)}\right).\end{aligned}$$

$$\begin{aligned}&\frac{d^k}{dx^k}\chi_{12}(x,\lambda)\\ &= -\frac{a_2^3 s}{2}\frac{d^k}{dx^k}\sin\frac{s(x-1)}{a_2} + \frac{a_1^3 s\delta_2}{4}\frac{d^k}{dx^k}\left(e^{\frac{s(x-1)}{a_2}} - e^{-\frac{s(x-1)}{a_2}}\right) + O\left(|s|^{k+1} e^{|s|\frac{(1-x)}{a_2}}\right).\end{aligned}$$

$$\begin{aligned}&\frac{d^k}{dx^k}\chi_{21}(x,\lambda)\\ &= -\frac{a_1^2 s^2\delta_1\chi_{22}'(0)}{2}\frac{d^k}{dx^k}\cos\frac{sx}{a_1} + \frac{a_1^3 s\delta_2\chi_{22}(0)}{2}\frac{d^k}{dx^k}\sin\frac{sx}{a_1}\\ &+\frac{a_1^2 s^2\delta_1\chi_{22}'(0)}{4}\frac{d^k}{dx^k}\left(e^{\frac{sx}{a_1}} + e^{-\frac{sx}{a_1}}\right) + \frac{a_1^3 s\delta_2\chi_{22}(0)}{4}\frac{d^k}{dx^k}\left(e^{\frac{sx}{a_1}} - e^{-\frac{sx}{a_1}}\right)\\ &+O\left(|s|^{k+2} e^{|s|\left(\frac{a_1-a_2x}{a_1a_2}\right)}\right).\end{aligned}$$

$$\begin{aligned}\frac{d^k}{dx^k}\chi_{22}(x,\lambda) &= -\frac{a_2^2 s^2}{2}\frac{d^k}{dx^k}\cos\frac{s(x-1)}{a_2} + \frac{a_2^2 s^2}{4}\frac{d^k}{dx^k}\left(e^{\frac{s(x-1)}{a_1}} - e^{-\frac{s(x-1)}{a_1}}\right)\\ &+O\left(|s|^{k+1} e^{|s|\frac{(1-x)}{a_2}}\right),\end{aligned}$$

where $k = \overline{0,3}$. Each of these asymptotic formulae holds uniformly for x.

Theorem 4.4. *Let $\lambda = s^4$, $s = \sigma + it$. Then the characteristic functions $W_i(\lambda)$ $(i = 1,2)$ have the following asymptotic formulae:*

$$W_1(\lambda) = W_2(\lambda) = O\left(|s|^{11} e^{2|s|\left(\frac{a_1+a_2}{a_1 a_2}\right)}\right).$$

Proof. Substituting the asymptotic equalities $\frac{d^k}{dx^k}\chi_{11}(-1,\lambda)$ and $\frac{d^k}{dx^k}\chi_{21}(-1,\lambda)$ into the representation of $W_1(\lambda)$, we get

$$\begin{aligned}
&W_1(\lambda)\\
&= \begin{vmatrix} 0 & 0 & \chi_{11}(-1,\lambda) & \chi_{21}(-1,\lambda) \\ 0 & \beta_2 & \chi'_{11}(-1,\lambda) & \chi'_{21}(-1,\lambda) \\ 0 & -\beta_1 & \chi''_{11}(-1,\lambda) & \chi''_{21}(-1,\lambda) \\ -1 & 0 & \chi'''_{11}(-1,\lambda) & \chi'''_{21}(-1,\lambda) \end{vmatrix} \\
&= \frac{a_1^5 \delta_1 \delta_2 s^3}{8} \left(\chi'_{12}(0)\chi_{22}(0) - \chi_{12}(0)\chi'_{22}(0)\right) \\
&\times \left(\begin{vmatrix} 0 & 0 & \cos\frac{s}{a_1} & e^{-\frac{s}{a_1}} - e^{\frac{s}{a_1}} \\ 0 & \beta_2 & -\frac{s}{a_1}\sin\frac{s}{a_1} & \frac{s}{a_1}\left(-e^{-\frac{s}{a_1}} - e^{\frac{s}{a_1}}\right) \\ 0 & -\beta_1 & -\frac{s^2}{a_1^2}\cos\frac{s}{a_1} & \frac{s^2}{a_1^2}\left(e^{\frac{s}{a_1}} - e^{\frac{s}{a_1}}\right) \\ -1 & 0 & -\frac{s^3}{a_1^3}\sin\frac{s}{a_1} & \frac{s^3}{a_1^3}\left(-e^{-\frac{s}{a_1}} - e^{\frac{s}{a_1}}\right) \end{vmatrix} \right) \\
&+ \begin{vmatrix} 1 & 0 & \sin\frac{s}{a_1} & e^{-\frac{s}{a_1}} + e^{\frac{s}{a_1}} \\ 0 & 0 & \frac{s}{a_1}\cos\frac{s}{a_1} & s\left(-e^{-\frac{s}{a_1}} + e^{\frac{s}{a_1}}\right) \\ 0 & -1 & -\frac{s^2}{a_1^2}\sin\frac{s}{a_1} & s^2\left(e^{\frac{s}{a_1}} + e^{\frac{s}{a_1}}\right) \\ 0 & 0 & -\frac{s^3}{a_1^3}\sin\frac{s}{a_1} & s^3\left(-e^{-\frac{s}{a_1}} + e^{\frac{s}{a_1}}\right) \end{vmatrix} \\
&+O\left(|s|^{15} e^{2|s|\left(\frac{a_1+a_2}{a_1 a_2}\right)}\right) \\
&= 0.
\end{aligned}$$

Analogically, we can obtain the asymptotic formulae of $W_2(\lambda)$. □

Corollary 4.5. *The real eigenvalues of the problem (1.1)-(1.9) are bounded below.*

Proof. Putting $s^2 = it^2$ $(t > 0)$ in the above formulas, it follows that

$$W\left(-t^2\right) \to \infty \text{ as } t \to \infty.$$

Therefore, $W(\lambda) \neq 0$ for λ negative and sufficiently large in modulus. □

Corollary 4.6. *The non-real eigenvalues of the problem (1.1)-(1.9) are bounded below and above.*

Now we can obtain the asymptotic approximation formulae for the eigenvalues of the considered problem (1.1)-(1.9).

Since the eigenvalues coincide with the zeros of the entire function $W(\lambda)$, it follows that they have no finite limit. Moreover, we know from Corollary 4.5 that all real eigenvalues are bounded below. Hence, we may renumber them as $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$, listed according to their multiplicity.

Theorem 4.7. *The eigenvalues $\lambda_n = s_n^4$, $n = 0, 1, 2, ...$ of the problem (1.1)-(1.9) have the following asymptotic formulae for $n \to \infty$:*

$$\sqrt[4]{\lambda_n'} = \frac{a_1\pi\,(2n-1)}{2} + O\left(\frac{1}{n}\right) \text{ and } \sqrt[4]{\lambda_n''} = \frac{a_2\pi\,(2n+1)}{2} + O\left(\frac{1}{n}\right).$$

Proof. By applying the well-known Rouché's theorem, which asserts that if $f(s)$ and $g(s)$ are analytic inside and on a closed contour C, and $|g(s)| < |f(s)|$ on C, then $f(s)$ and $f(s) + g(s)$ have the same number zeros inside C provided that each zero is counted according to their multiplicity, we can obtain these conclusions. □

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